Deconvolving oscillatory transients with a Kalman filter

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Abstract

This paper describes a method to filter oscillatory transients from measurements of a time series which were at least an order of magnitude larger than the signal to be measured. Based on a Kalman filter, it has an optimality property and a natural scaling parameter that allows to tune it to high resolution or low noise.

1 How transients can spoil measurements

In her Matura thesis [5], Carmen Müller tried to measure the opening shock of small parachutes in the Ackeret wind tunnel of the Federal Institute of Technology in Zürich (ETHZ). A specially constructed mounting tube ejected the parachute using a piston powered by a small black powder charge into the air stream. While this accurately reflected the way parachutes are usually ejected from small rockets, it also induced heavy oscillations in the measurement equipment and probably also excited waves within the wind tunnel. As a result, the data was almost unusable. The large amplitude transients were at least one order of magnitude larger than the effects to be measured (figure 1).

Upon inspection of the data, two oscillatory components with frequencies of about 40Hz and 12.2Hz were found. Assuming that the measured signal is a superposition of the actual force to be measured $f(t)$ and a transient of period T, convolution with a function of constant value T^{-1} on the interval $[-T, 0]$, or equivalently a moving average over the samples that fit in an interval of length T, eliminated the transient. In this way, the two dominant transients could be removed, and an approximation to the opening shock force was obtained. The

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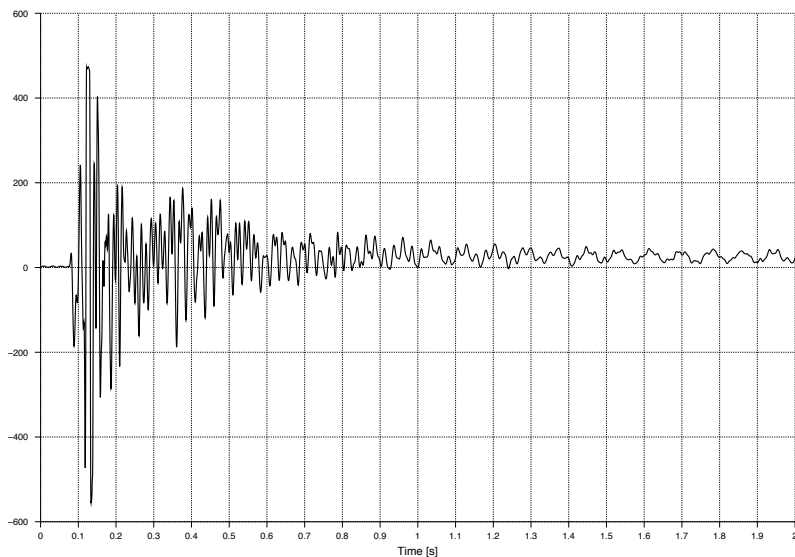
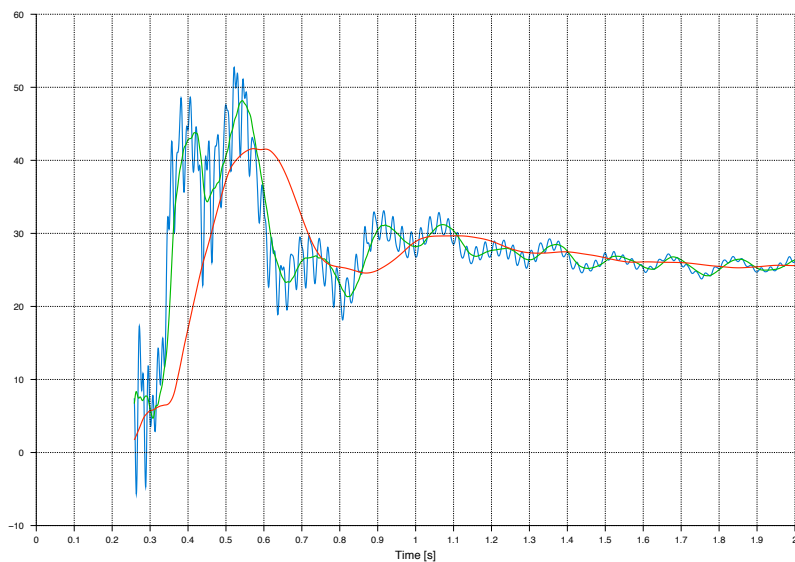Figure 1: Wind tunnel measurements of opening shock dominated by transients

Figure 2: Opening shock measurements filtered with moving averages of different lenghts. The red curve obviously does not accurately represent the forces during parachute opening.

price to pay was of course that the signal f was also modified by the convolution. The modifications were large enough that it became unreasonable to draw any quantitative conclusions from the measurements (figure 2).

Using an exponential weighting function, it was possible to slightly improve the filtering, taking into account the damping of the transients. Since the carrier of the convolution of the intervals of length 24ms and 83ms corresponding to the frequencies of the transients is as an interval of at least 107ms, the convolution reduced the effective resolution of the measurements to less than 10 samples per second, which is of the same order of magnitude as the force changes during the opening shock.

While this approach was successfull und fully adequate for a Matura thesis, the question was still open whether a better filtering method could reveal more details about the opening shock. Ideally, such a filter would have a scaling parameter that allows to tune it to different resolutions, where of course fine grained resolution comes at the price of additional noise and incomplete filtering of the observed transients. Below we show that a Kalman filter is ideally suited for this situation, and that the system errors represent a set of parameters that allow this kind of tuning of the filter.

2 How to get rid of transients

In the construction below, we will not develop the Kalman filter equations, but only indicate the system equations from which the filter can be derived. Our notation closely follows standard texts on the discrete Kalman filter like [3] or [4].

2.1 The harmonic oscillator

The measurement system in the wind tunnel works as follows. The forces on the parachute and the ejection tube deflect the mounting hardware by a tiny fraction measured by piezoelectric sensors. So the mass m of the mounting hardware is subject to two forces, the forces we want to measure and the elastic forces, and we measure its position x as a function of time. If D is Hook's constant for the elastic forces, x obeys the differential equation

$$m\ddot{x} + \alpha\dot{x} + Dx = f, \tag{1}$$

where f is the force we want to measure. In a pseudostatic situation, we can assume that $\dot{x} = \ddot{x} = 0$, and so the force is proportional to the deflection value we measure. In the dynamic situation, we have to infer $f(t)$ from $x(t)$ as a function of time.

A naive approach would use the measurements of x and compute approximations of $\dot{x}$ and $\ddot{x}$ as finite difference quotients. Then an approximation for f can be computed from (1). However, numerical differentiation tends to be noisy, and the second derivative will probably be almost useless. The problem gets even more serious in light of the fact that in real applications, we need to filter several times, so we need a method to estimate f that does so with a controlled level of noise.

This is the classical setting for a discrete Kalman filter. At any point in time we would like the best estimate of the current position and speed of the mass m and the force f based on measurements of x at discrete points $n\Delta t$ in time. We use $x_n = (x(n\Delta t), \dot{x}(n\Delta t), f(n\Delta t))$ as the state vector of a linear system. The time evolution is given by the differential equation (1), which can be written, using the familiar substitution $v(t) = \dot{x}(t)$ in vector form as

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{D}{m} & -\frac{\alpha}{m} & \frac{1}{m} \end{pmatrix}\begin{pmatrix} x(t) \\ v(t) \\ f(t) \end{pmatrix} \tag{2}$$

Since we don't know anything about f, we assume for the purposes of the Kalman filter, that f is constant, i. e. $\dot{f} = 0$. Using the state vector $s = (x, v, f)$, the differential equation becomes

$$\frac{d}{dt}s = As$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{D}{m} & -\frac{\alpha}{m} & \frac{1}{m} \\ 0 & 0 & 0 \end{pmatrix}$$

Time evolution for a short time interval Δt is given by the matrix exponential

$$\varphi = e^{A\Delta t}.$$

The measurement equations are also very simple. Since we only measure the position, we can use the measurement matrix

$$H = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$

To completely define the Kalman filter, we have to specify the system and measurement errors. The measurement error is a well known characteristic of the measurement equipment. The system errors for x and v have to be kept very small, as we believe the harmonic oscillator equation is followed quite accurately. In contrast, the evolution equation for the force is only a very rough approximation. We attribute almost all deviations of the real system from the model to external forces. Thus the flucutations due to measurement errors are the only sources of
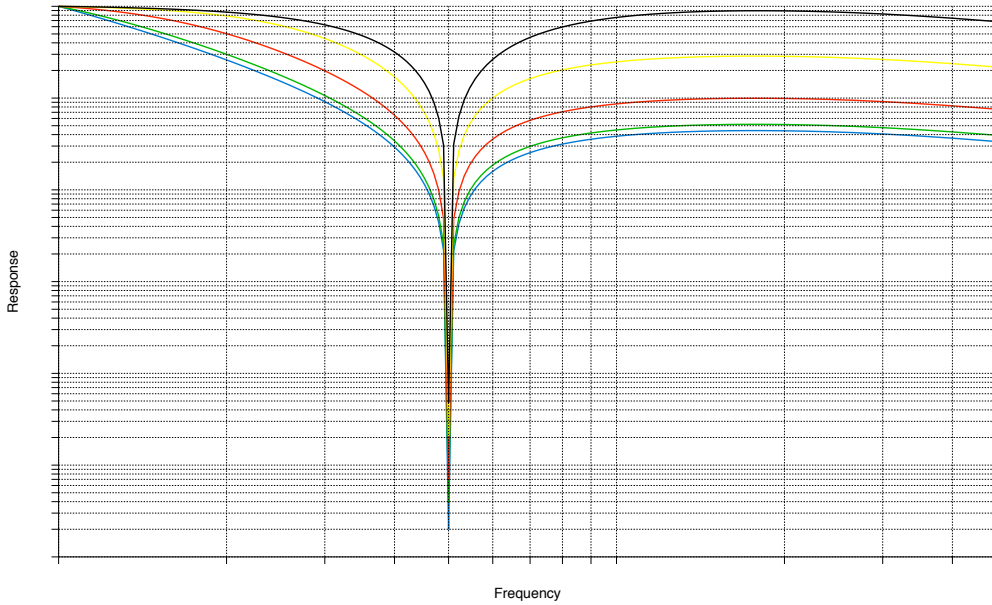
Figure 3: Frequency response of Kalman filter based transient filter for various values of σ_f^2

errors for x and v, while for f we have to accept errors as large as the slope of f times the time step. If σ_x^2 is the position measurement variance and σ_f^2 the variance of the force, then a reasonable approximation of the system error covariance matrix Q and the measurement error covariance R is

$$Q = \begin{pmatrix} \sigma_x^2 & 0 & 0 \\ 0 & \sigma_x^2 & 0 \\ 0 & 0 & \sigma_f^2 \end{pmatrix}, \qquad R = \begin{pmatrix} \sigma_x^2 \end{pmatrix}.$$

In the example of the wind tunnel measurements of the opening shock, the measurement error was about $0.001\mathrm{N}^2$, while the force error was on the order of $10^5\mathrm{N}^2$.

Using a large value of σ_f^2 causes the Kalman filter to "trust" the measurements and agressively attribute deviations from the evolution given by the differential equation to changing external forces. Of course, f will be much more noisy in this case, since measurement noise is converted to a large degree to noise in f. If σ_f^2 is small, then the filter tries not to modify f too quickly, and attributes deviations from the differential equation to a larger extent to the measurement errors. The resulting f will be much smoother, but steep changes in f will be filtered away. Thus σ_f^2 is a natural scaling parameter we asked for in the introduction.

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2.2 Frequency response

Figure 3 shows the numerically computed frequency response for various values of σ_f^2. As expected, the notch at 50Hz is very pronounced. For large enough values of σ_f^2, there is hardly any loss for high frequencies. For small values of σ_f^2, however, the loss is much larger, rather similar to the $\sin x/x$ characteristic of the moving average filter.

2.3 Implementation considerations

2.3.1 Building the filter from ω

None of the coefficients m, α and D of the differential equation are known. Only the frequency and the damping of the transients can be measured. The characteristic equation of the harmonic oscillator equation

$$\omega^2 + \frac{\alpha}{m}\omega + \frac{D}{m} = 0 \tag{3}$$

has two complex conjugate roots ω and $\bar{\omega}$. To simplify the notation, we write $a = \frac{\alpha}{D}$ and $b = \frac{D}{m}$. We conclude that

$$\mathrm{Re}\,\omega = \frac{a}{2} \quad \text{and} \quad \mathrm{Im}\,\omega = \sqrt{b - \frac{a^2}{4}}.$$

Given ω, the differential equation can be obtained from

$$a = 2\,\mathrm{Re}\,\omega \quad \text{and} \quad b = (\mathrm{Im}\,\omega)^2 + \frac{a^2}{4}.$$

This allows to easily construct the Kalman filter for the oscillator from frequencies obtained e. g. by spectral analysis.

2.3.2 Precomputing the Kalman gain matrix

It is well known that when all the matrices defining the Kalman filter are constant, then the Kalman gain matrix converges. For given values of the system and measurement errors, the Kalman gain matrix can thus be computed beforehand and "hard wired" into the filter.

2.4 Coupled oscillators

If a system exhibits several oscillations, we can still assume that the system is linear, and that each oscillation corresponds to a harmonic oscillator with that
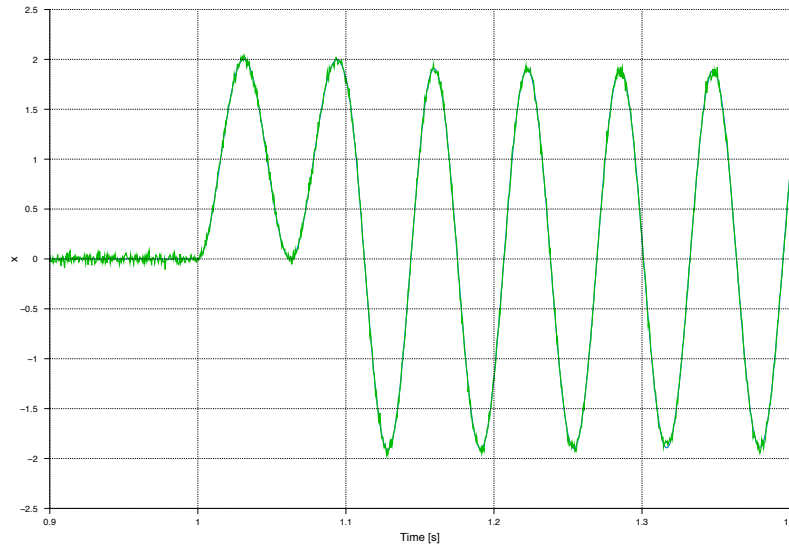
Figure 4: Harmonic oscillator excited by a pulse of 0.1s duration at time $t = 1$.

frequency. Such a system can be described by a higher dimensional state vector, and it is possible to construct a Kalman filter for it. However, it would become rather large, and the following simpler approach is preferable.

Each one of the coupled oscillators is accurately modelled by a harmonic oscillator on which two external forces act, the force f and the combined forces excerted by the other oscillators. The latter have a well known frequency, so filtering the signal in turn for all the frequencies of transients identified using the method described above removes all the transients from the signal f.

3 Success stories

The method described in the previous section was first applied to synthetic data to verify that the filter is indeed capable of retrieving the synthetic signal. Then the data was applied to the opening shock data that originally motivated this research, and to a related problem involving thrust curves of rocket motors.

3.1 Test data

To verify the properties of the Kalman filter based deconvolution of transients, a solution for the harmonic oscillator equation
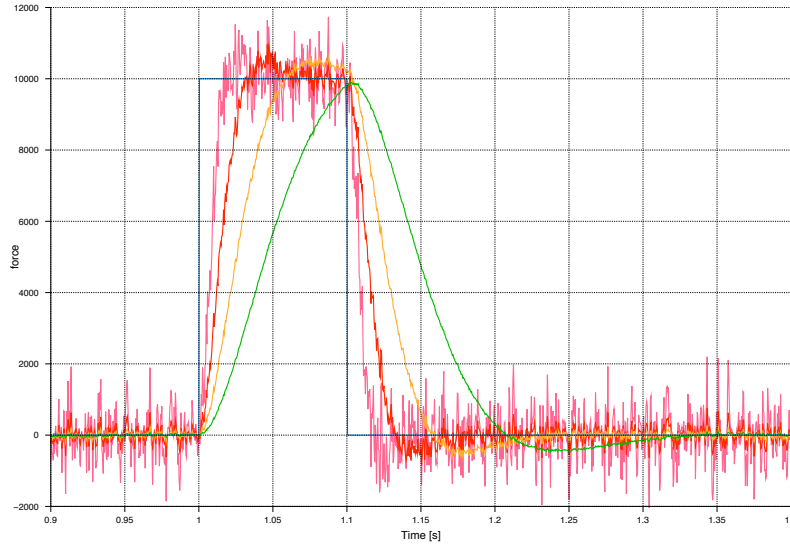
$$\ddot{x} + 0.1 \cdot \dot{x} + 1000 \cdot x = f(t)$$

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Figure 5: Signal from figure 4 filtered by a Kalman filter for increasing σ_f^2, compared to the original excitation (blue curve).

was computed numerically for the force function

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f(t) = \begin{cases} 0 & t < 1 \vee t > 1.1 \\ 10000 & t \in [1, 1.1] \end{cases},
$$

and some gaussian noise with $\sigma = 0.1$ added to simulate the measurement errors. The resulting measurement data is shown in figure 4. Figure 5 shows the output of a Kalman filter based deconvolution with $\sigma_f^2 = 10^k$, $k = 3, 4, 5, 6$. As expected, noise increases with increasing σ_f^2. However, a simple moving average over just a few samples can remove large parts of the noise without distorting the signal very much.

3.2 Opening shock of parachutes

As a practical test case, the filtering method was applied to the opening shock data. A chain of three kalman filters was constructed using the following algorithm.

1. Take the raw measurement data as the current signal to filter.

2. Find the dominant frequency ω in the current signal.

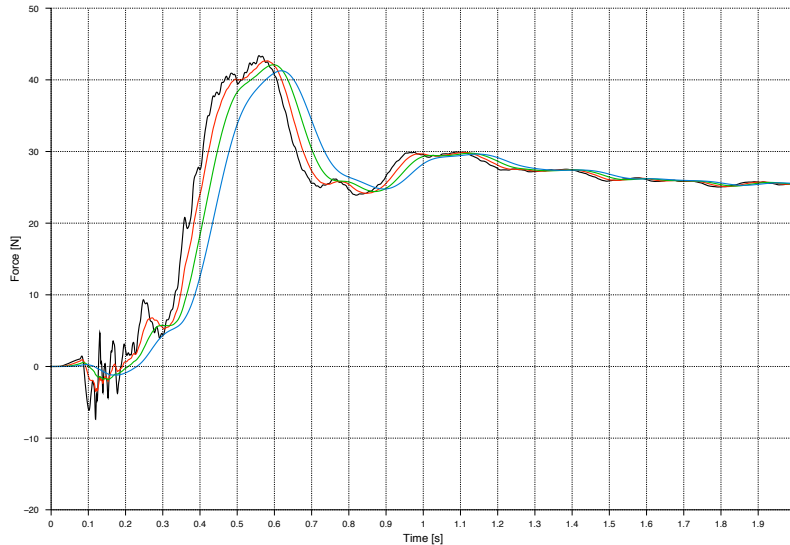3. Construct a Kalman filter for frequency ω.

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Figure 6: Opening shock measurements from figure 1 filtered with three Kalman filters for different values of the force error variance σ_f^2.

4. Filter the current signal, increase σ_f^2 until a new dominant oscillation appears, in which case you continue at step 2, or the signal becomes implausible.

5. Reduce σ_f^2 until you again get a physically plausible signal.

During this process, in addition to the frequencies already found in [5], an additional oscillation at 85.7Hz was found. The result of the filter for different values of σ_f^2 is shown in figure 6. The curves following most closely the physical force signal were obtained with $\sigma_f^2 = 100000\text{N}^2$ and $\sigma_f^2 = 200000\text{N}^2$, the second again shows some probably unphysical transients.

The reconstruction of the signal using the Kalman filter at least allows now to draw some quantitative conclusions from the opening shock data. E. g. it allowed to confirm the empirical rule used in some circles that the opening shock is about twice as strong as the drag of the fully inflated parachute.

3.3 Thrust curve of high thrust rocket motors

Aerotech Consumer Aerospace produces rocket motors for hobby rockets and small research rockets. For each motor, thrust curves are available from various
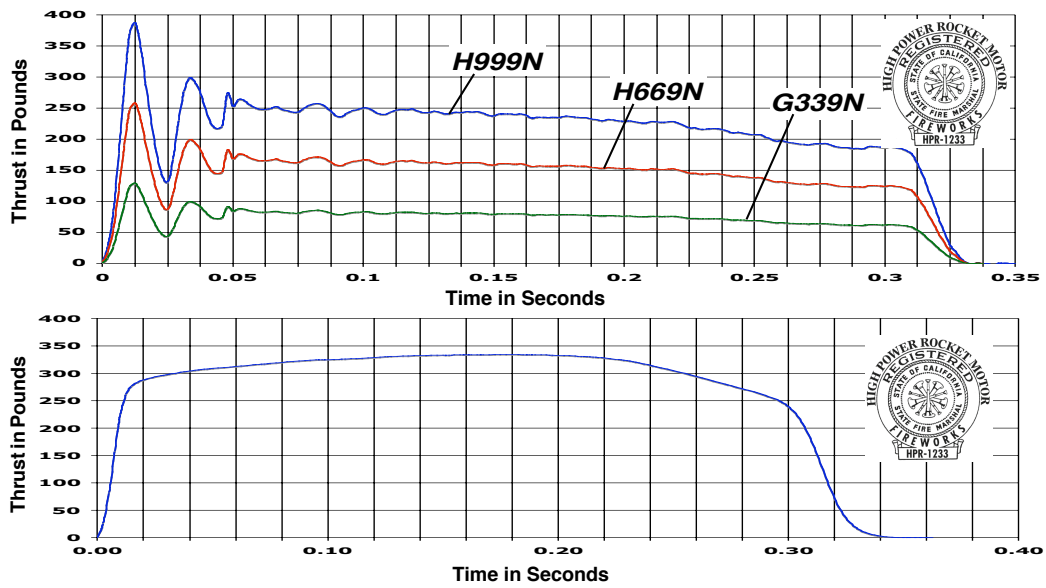
Figure 7: Thrust curves of 38mm Warp-9 propellant rocket motors as they appear on the instruction leaflet accompanying the reload kits. H999 top curve in top diagram, I1299 in bottom diagram.
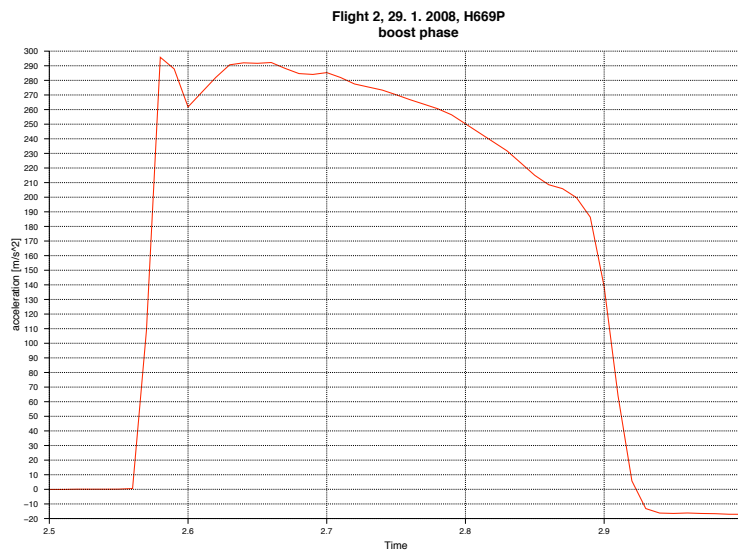

Figure 8: Acceleration profile of a rocket flight on a H669 motor, no oscillations
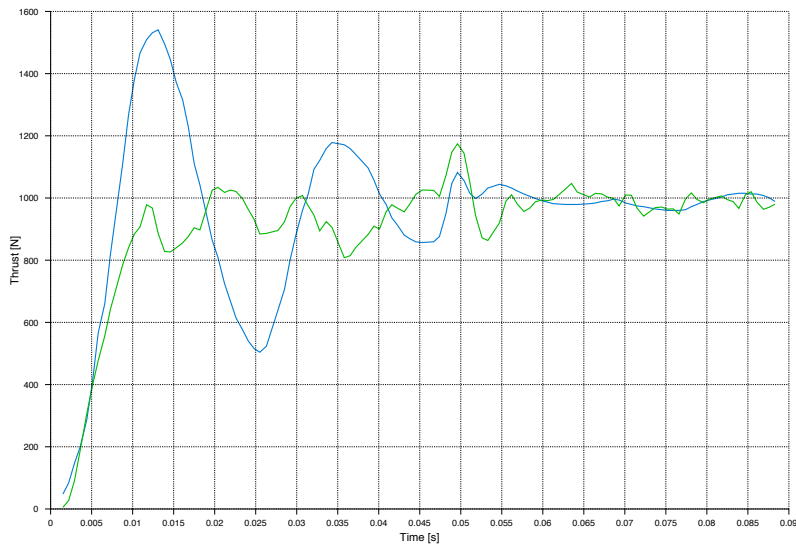
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Figure 9: Original thrust curve (blue) and version filtered for transients using a Kalman filter (green)

sources on the Internet, and also from Aerotech. The more recent motors using on the Warp-9 propellant have extremely high thrust for a very short time interval. The motor H999 has average thrust of 999N during only 0.3s. The thrust curve shown in figure 7 bottom (from [1]) shows an initial oscillation which cannot be explained by any known mechanism in solid propellant motors. As a matter of fact, measuring the acceleration during an actual rocket flight with this motor, as was done with the high precision altimeter developed during the rocketware project (http://rocketware.othello.ch), shows no such oscillation (see figure 8). Also the only slightly larger Motor I1299 (figure 7 bottom, from [2]) has a completely flat thrust curve. So most probably the oscillation is an artefact of the measuring equipment used for these curves. Of course, measuring thrust of a rocket motor is similar to measuring aerodynamic forces in a wind tunnel. Thrust acts on the motor, which is displaced slightly, the displacement is proportional to the force. But without sufficient damping, the test stand holding the motor may oscillate adding an oscillatory transient.

To corroborate this hypothesis, the Kalman filter approach was applied to a manually digitized version of the thrust curve. Figure 9 shows the filtered version which is obviously much closer to the thrust curve measured by the rocketware accelerometer. The remaining noise was probably introduced when manually reading thrust values from the printed thrust curve.

It turns out that the thrust curves were in fact measured on different test stands.

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Thrust curves for I1299 were obtained on Aerotech's test stand, the test stand of the Tripoli Rocketry Association was used for motors that show strong oscillations.

4   Conclusion

The results of actual measurement data filtered using the Kalman filter approach show that oscillatory transients of at last the magnitude of the signal can be effectively filtered to reveal the signal. The measurements must be precise enough to render the full dynamic range, otherwise the noise introduced by the filter will quickly become larger than the signal. If this condition is satisfied, however, the method can be expected to be close to optimal, as the Kalman filter has such a property. If better models for f are available, it is possible that the method could be refined to yield even better results.

References

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[2] Aerotech Consumer Aerospace, *RMS 38/480 Warp-9 Reload Kit I1299N-P*, downloaded from http://www.aerotech-rocketry.com/customersite/resource-_library/Instructions/HP-RMS_Instructions/38_480n_in_20069-1.pdf on march 22, 2008.

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